Heterogeneous Population, Economic Growth and Income Distribution

Nissim Ben David¹, Uri Ben-Zion²

Abstract
We present an overlapping generation model with an example and a simulation of an economy that consists of three classes of workers. Agents might invest in education or capital. As people are more educated, their population growth rate is lower. The poorest class cannot borrow in order to invest in education or capital and its population is growing faster. As capital grows, educated agents can be mobilized into a state of firm owners. We show that under such conditions, while the wage per worker of the poorest class can increase, income distribution worsens as the economy grows.

Key Words: Heterogeneous Population, Economic Growth, Income Distribution

1. Introduction
Many countries around the world have a heterogeneous population consisting of several ethnic and cultural groups. Variety of population is usually accompanied by very large differences in population growth rates for different classes, leading to drastic changes in population and income distribution.

In this paper, we present a theoretical model with an example and a simulation of an economy that consist of three classes, where the lowest economic class has a higher population growth rate. Our aim is to suggest a government policy that would reduce population growth by taxing the richer classes and allocating budgets to activities that lead to a reduction in the population growth rate of the poor. Such a step would improve income distribution, as well as raise the wage per worker in the lowest class along the growth path of the economy. Delaying such a policy would propel economies with a heterogeneous population into a path with a much larger poor class, and a much worse income distribution.

The relationship between population and economic growth is complex and the historical evidence is ambiguous, particularly concerning causes and impacts (Thirlwall 1994, p. 143). Becker, Glaeser, and Murphy (1999, p. 149) demonstrated in a theoretical model that large population growth could have both negative and positive impacts on productivity. A large population may reduce productivity because of diminishing returns from more intensive use of land and other natural resources. Conversely, a large population could encourage greater specialization, and enlarged markets would increase returns to human capital and knowledge. Thus, the net relationship between greater population and economic growth depends on whether the inducements to human capital and knowledge expansion are stronger than the diminishing returns on natural resources. Therefore, it is important to examine the population and economic growth nexus.

Rapid population growth tends to depress per capita savings and retard the growth of physical capital per worker. The need for social infrastructure is also broadened and public expenditures must be absorbed in providing for the needs of a larger population rather than in providing productive assets directly (Meier 1995, pp. 276-77).

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The economic literature discusses various determinants of economic growth. Many researchers focus on human capital accumulation and technology diffusion (see, for example, Creedy and Gemmell (2005), Aiginger (2005) and Zagler (2005)).

Empirical evidence indicates that, except for very poor countries or households, increases in per capita income tend to reduce fertility. These studies have typically found important linkage between economic variables - such as per capita income, wage rates, level of female and male education, and urbanization - to fertility and mortality (See Wahl (1985), Schultz (1989) and Barro and Lee (1994). Due to these empirical findings, income inequality has been connected to population growth rates.

Xavier and Martini (2006) found a spectacular reduction in worldwide poverty. Behind the reduction in poverty indexes, hides the uneven performance of various regions in the world. East and south Asia account for a large fraction of the success. Africa, on the other hand, seems to have moved in the opposite direction. The dismal growth performance of the African continent has meant that poverty rates and head counts increased substantially over the last three decades. The implication is that where poverty was mostly an Asian phenomenon 30 years ago (80 percent of the world’s poor lived in east and south Asia), poverty is, today, an essentially African problem (75 percent of the poor live in Africa today, whereas only 19 percent live in Asia). According to Xavier and Martini (2006), the Gini coefficient for the world distribution of income reveals that when China is excluded from the analysis, worldwide individual income inequalities increase from 0.620 to 0.648, an overall increase of 4.4 percent.

Table: a present’s data of population growth rate and income per capita for various regions around the world.

<table>
<thead>
<tr>
<th>Region</th>
<th>Population Growth rate</th>
<th>GDP Per Capita</th>
</tr>
</thead>
<tbody>
<tr>
<td>World</td>
<td>1.2</td>
<td>7,448</td>
</tr>
<tr>
<td>Least Developed Countries</td>
<td>2.4</td>
<td>431</td>
</tr>
<tr>
<td>South Asia</td>
<td>1.6</td>
<td>768</td>
</tr>
<tr>
<td>Middle East and North Africa</td>
<td>1.8</td>
<td>2,507</td>
</tr>
<tr>
<td>Sub-Saharan Africa</td>
<td>2.5</td>
<td>829</td>
</tr>
<tr>
<td>India</td>
<td>1.4</td>
<td>820</td>
</tr>
<tr>
<td>China</td>
<td>0.6</td>
<td>2,000</td>
</tr>
<tr>
<td>East Asia and Pacific</td>
<td>0.8</td>
<td>4,359</td>
</tr>
<tr>
<td>European Monetary Union</td>
<td>0.5</td>
<td>34,307</td>
</tr>
<tr>
<td>Lower-Middle Income Countries</td>
<td>0.9</td>
<td>2,038</td>
</tr>
<tr>
<td>Middle Income Countries</td>
<td>0.8</td>
<td>3,053</td>
</tr>
<tr>
<td>Upper-Middle Income Countries</td>
<td>0.8</td>
<td>5,913</td>
</tr>
<tr>
<td>High Income Countries</td>
<td>0.7</td>
<td>36,608</td>
</tr>
<tr>
<td>High Income OECD Countries</td>
<td>0.6</td>
<td>38,190</td>
</tr>
</tbody>
</table>

Source: World Bank Development Data & Statistics
http://go.worldbank.org/1SF48T40L0

We can see a clear negative relation between population growth rate and income per capita.

We should also notice that population growth rate in China is very low (while per capita income has been rising very fast during last decade).

In Appendix 1, we present the United Nations data for population size, distribution and yearly growth rates for the years 1750 – 1999 as well as forecasted size for the years 2050 and 2150. We can see that the relative size of poor countries’ populations is constantly rising and is expected to keep rising during the upcoming 150 years, while Europe and North America's relative share in world population is constantly
decreasing. (In 1750, Europe and North America compromised about 21% of world’s population, in 1999
about 17%, and it is expected to be about 9% of the world's population in 2150)

Population changes will most probably have a real impact on economic growth, and therefore, the
development of sound models will be increasingly relevant.

Several researchers have introduced country-specific population growth rates. For example, Sayan
(2005) investigated the implications of the addition of differential population dynamics in two regions
assumed to be identical in every respect except for the way their populations evolved over time in a context
of a model of international trade within an Overlapping Generations framework. Naito and Zhao (2008)
formulated a two-country, two-good, two-factor, two-period-lived overlapping generations model to examine
how population aging determines the pattern of gains from trade. Yakita (2012) examined the effects of
different aging speeds on international trade patterns in an open Overlapping Generations model. However,
we couldn’t find papers allowing for different growth rates for different sections of society supplying
different factors of production.

In this article, we extended the Over-Lapping Generations model with capital based on Diamond (1965)
to include three types of individuals. Each individual belongs to a certain class of workers, according to
his/her production capability. The population growth rate is highest for the lowest class due to lack of
education and other social factors.

In Diamond's model (1965), all individuals are similar and live during two periods. An individual
supplies one unit of labor when young, for which s/he receives a wage, and zero units when old. His/her
consumption when old is financed by savings done when young in addition to the return on those savings.
The savings of each generation finance the investment of the next generation. A connection is formed
between current capital and future capital. Capital converges to a long-term stationary level.

All firms use the same technology and each firm employs workers from the two worker categories and
pay all the workers in the same class equal wages. Workers in different categories are paid different wages.
The aggregate savings of each generation finance investment in capital and in the education of the next
generation. Increasing the level of capital would cause an increase in the number of firms and firm owners as
well. The new firm owners would come from a new generation born in the firm owners' class while others
would come from the higher working classes. The mobility of workers, following the growth in capital,
would change income distribution. In each generation, wage distribution can be calculated so that it can be
traced on the path to steady state equilibrium.

This paper is organized in the following manner. The model structure is laid down and competitive
equilibrium conditions are presented in section II. A specific example
is presented in Section III. To confirm the results of sections II and III, a simulation
is presented in section IV. Section V presents the summary.

2. Model Structure

Consider a perfectly competitive world where economic activity is performed over infinite discrete time,
$\tau = 1, 2, \ldots, \infty$. There are many firms and each firm utilizes two factors, labor and capital, in the production
process.

All individuals live during two periods of time. An individual works during the first period and retires
at the beginning of the second. Each individual saves part of his/her first period income so that the savings,
including the return, finance his/her second period consumption. During each period, both young and old
people are alive.
Supply and Demand for Capital

In each generation, young entrepreneurs are without means and must borrow in order to finance the capital that is needed by their new firms. The younger generation borrows from the older generation an amount that equals the value of the depreciated capital of the last generation plus a share $\lambda_i$ of the savings of the last generation. Each generation would invest in human capital a share $(1 - \lambda_i)$ of the savings.

The capital stock at time $t-1$ is $K_{t-1}$, the total savings of the last generation is $S_{t-1}$, depreciation rate is $d$ and $I_t$ is the net investment at period $t$.

In generation $t$, the young would borrow from the old an amount of $(1-d)K_{t-1} + \lambda_i S_{t-1}$. We get that the capital stock at time $t$ would be:

$$K_t = (1-d)K_{t-1} + I_t = (1-d)K_{t-1} + \lambda_i S_{t-1}$$ (1)

(Notice that $\lambda S_{t-1} = I_t$, where $\lambda_i$ is the share of savings invested in capital).

The Sources of Financing the Capital

At the beginning of period $t$, younger generation entrepreneurs would borrow from the older generation an amount equal to $K_t$. We assume that the old generation is paid zero real interest rate for the loan.

The current younger generation would repay the loan at the end of period $t$. Assuming that income of the young generation is $Y_t$ and that their savings are $S_t$ we get that consumption of the young generation in period $t$, $C_t$, is:

$$C_t = Y_t - [(1-d)K_{t-1} + \lambda_i S_{t-1}] - S_t$$ (2)

While consumption of the old generation is

$$C^0_t = [(1-d)K_{t-1} + \lambda_i S_{t-1}]$$ (2a)

Assuming that investment in real capital is $I_t$ while investment in human capital is $E_t$.

We get that aggregate demand in period $t$ is:

$$C_t + C^0_t + I_t + E_t = \{Y_t - [(1-d)K_{t-1} + \lambda_i S_{t-1}] - S_t\} + \{[(1-d)K_{t-1} + \lambda_i S_{t-1}]\} + I_t + E_t$$

$$= Y_t \rightarrow I_{t+1} + E_{t+1} = S_t = \lambda_i S_t + (1-\lambda_i)S_t$$ (2b)

Where $I_t = \lambda_i S_t$.

We should notice that current young generation repay the loan to the old generation and at the end of period $t$ they will lend to the new young generation born at $t+1$ an amount of $(1-d)K_t + \lambda_{t+1} S_t$, which is the amount of capital in period $t+1$.

Production

$L_{i0} \quad \text{for}\quad i=0,1,2$ - Represents the number of agents in class $i$

$L_{01}$ is the number of entrepreneurs and each firm has one firm owner, such that $L_{01}$ is also the number of identical active firms at the market in period $t$.

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3Selling depreciated capital and lending savings to the young enables the old generation to smooth consumption during their life span. The old generation is willing to give the loan at zero interest rate due to decreasing marginal utility from consumption in each time period.
$L_{ij}$ is the number of workers in class 1.

$L_{2j}$ is the number of workers in class 2.

$E_i -$ is total years of education of workers belonging to class $i$.

We assume that each firm produces according to the Cobb–Douglas constant return to scale function.

The output produced by one firm at time $t$ is:

$$Y^j_t = k^j_t a L_{it}^\beta L_{2t}^\beta E_{0t}^\gamma E_{1t}^\gamma E_{2t}^\gamma$$

(3)

Where: $\beta_1 < \beta_2$ and $\sum_{i=0}^2 \beta_i + \sum_{i=0}^2 \gamma_i + \alpha = 1$ .

$k_t$ - is the amount of capital in firm $j$ and each firm employs the same optimal amount of capital. The amount of capital of each firm is

$$k_t = \frac{K}{L_{0t}}$$

(4)

Where total amount of capital in the economy is $K_t$

**Income Distribution**

The product, $Y^j_t$, in each firm $j$ is divided between the entrepreneurs and the workers. If labor is perfectly mobile, then workers in each class will get their marginal product. Workers’ marginal product is equal to additional production due to an additional unit of labor plus marginal product due to an additional amount of education used by the firm when it employs an additional worker$^4$, so that:

$$w_{it} = \beta_1 k^j_t a L_{it}^\beta L_{2t}^{\beta-1} E_{0t}^\gamma E_{1t}^\gamma E_{2t}^\gamma + \gamma_1 K^j_t a L_{it}^\beta L_{2t}^\beta E_{0t}^\gamma E_{1t}^{\gamma-1} E_{2t}^\gamma$$

$$w_{2t} = \beta_2 k^j_t a L_{it}^\beta L_{2t}^{\beta-1} E_{0t}^\gamma E_{1t}^\gamma E_{2t}^\gamma + \gamma_2 K^j_t a L_{it}^\beta L_{2t}^\beta E_{0t}^\gamma E_{1t}^{\gamma-1} E_{2t}^\gamma$$

(5a)

Where $w_{it}$ is the wage of a worker in class $i$ (each worker’s wage consists of marginal product of labor plus marginal product of education).

For each firm, we get that total income of the workers in class 1 is:

$$L_o MPL_o + E_o * MPE_o = \beta k^j_t a L_{it}^\beta L_{2t}^{\beta-1} E_{0t}^\gamma E_{1t}^\gamma E_{2t}^\gamma * L_o + \gamma k^j_t a L_{it}^\beta L_{2t}^\beta E_{0t}^\gamma E_{1t}^{\gamma-1} E_{2t}^\gamma * E_o = (\beta_1 + \gamma_1) Y^j_t$$

(5b)

Where $MPE_{it}$ is the marginal product of human capital of a worker belonging to class 1, while the total income of the workers in class 2 is:

$$L_{2o} MPL_{2o} + E_{2o} * MPE_{2o} = \beta k^j_t a L_{it}^\beta L_{2t}^{\beta-1} E_{0t}^\gamma E_{1t}^\gamma E_{2t}^\gamma * L_o + \gamma k^j_t a L_{it}^\beta L_{2t}^\beta E_{0t}^\gamma E_{1t}^{\gamma-1} E_{2t}^\gamma * E_o = (\beta_2 + \gamma_2) Y^j_t$$

(5c)

The income of the entrepreneur is:

$$W_{0t} = k^j_t a L_{it}^\beta L_{2t}^\beta E_{0t}^\gamma E_{1t}^\gamma E_{2t}^\gamma - (\beta_1 + \gamma_1 + \beta_2 + \gamma_2) Y_t = (1 - \beta_1 - \gamma_1 - \beta_2 - \gamma_2) Y^j_t$$

(6)

**Distribution of Total Production**

The total production of the economy $Y_t = L_{0t} Y^j_t$ is divided between the workers and the entrepreneurs.

The lowest income class of workers would get a total income of $L_{0t} (\beta_2 + \gamma_2) Y^j_t$, the second class
would get \( L_{00}(\beta_1 + \gamma_1)Y^1 \), while the total income of the entrepreneurs would be
\[ L_{00}(1 - \beta_1 - \gamma_1 - \beta_2 - \gamma_2)Y^1. \]

As we can see, given the level of capital, the production technology and the number of workers in each class, the total production in the economy is divided between the workers, the entrepreneur and the old generation and the Gini index can be calculated.

**Utility**

Defining \( C_{0i} \) - as consumption of agent i when s/he is young while \( C_{1i} \) is consumption of agent i when s/he is old, let us assume that the utility function of each agent in the economy, \( U = \ln(C_{0i}) + \gamma \ln(C_{1i}) \), is similar to agents belonging to all categories.

Each agent tries to maximize utility, subject to his/her budget constraints:
\[
\text{Max} \quad \ln(C_{0i}) + \gamma \ln(C_{1i})
\]
\[
s.t. \quad C_{oi} + \frac{C_{1i}}{1 + \gamma} = W_{it}
\]

Solving the problem\(^5\), we get:
\[
(8) \quad C_{oi} = \frac{1}{1 + \gamma} W_{it}
\]
\[
(9) \quad S_{i} = \frac{\gamma}{1 + \gamma} W_{it}
\]

Notice that agents who acquired education use part of their savings to finance the acquired education.

The aggregate savings in the economy would be:
\[
S_i = L_{0i} \sum_{a=0}^{1} S_{i} L_a = L_{0i} \sum_{a=0}^{1} \frac{\gamma}{1 + \gamma} W_{it} L_{at}
\]

Substituting \( Y^1 = \sum_{i=0}^{2} W_{it} L_{at} \) into (10) we get:
\[
(11) \quad S_i = L_{0i} \sum_{a=0}^{1} S_{i} L_a = \frac{\gamma}{1 + \gamma} L_{0i} Y^1, \quad \gamma \sum_{a=0}^{1} \frac{\gamma}{1 + \gamma} = \frac{\gamma}{1 + \gamma} Y^1,
\]

**Education**

The education of an agent belonging to classes 0 and 1 is determined by the rules
\[ (12) \quad MP_{E0} = MP_k \quad \text{and} \quad (12a) \quad MP_{E1} = MP_k. \]

\[ (13) \quad \gamma_0 k^a \gamma_1 \beta_1 L_{\gamma_1} \beta_2 L_{\gamma_2} E_{\gamma_2} E_{\gamma_1} E = \alpha k^a L_{\gamma_2} \beta_2 E_{\gamma_2} E_{\gamma_1} E_{\gamma_1} E_{\gamma_2} E_{\gamma_2} E_{\gamma_2}
\]

From (12) we get:
\[
\frac{\gamma_0 Y^1}{\alpha k_i} \rightarrow E_{0i} = \frac{\gamma_0 k_i}{\alpha}
\]

and from (12a) we get:
\[
\frac{\gamma_1 Y^1}{\alpha k_i} \rightarrow E_{1i} = \frac{\gamma_1 k_i}{\alpha}
\]

\(^5\)Notice that for \( r=0 \), the solution is not changed.
We assume an initial fixed education level for class 2. Due to their low income, agents of class 2 do not invest in education ($\gamma_2 = 0$) or capital due to inability to borrow.

Let assume that the cost of 1 year of education is fixed and equals to $P_E = 1$, then the total amount invested in education is:

$$
(1 - \lambda_t)S_t = L_{0t}(E_{0t} + E_{1t} + E_{2t} + E_{3t}) = L_{0t} \left( \frac{(\gamma_0 + \gamma_1)(L_{0t} + L_{1t})}{\alpha} \right)
$$

### Growth Path

According to equation 1, the amount of capital in period $t+1$ is:

$$
K_{t+1} = (1 - d)K_t + I_t 
$$

The net investment is financed by the savings of the current generation, so that $\lambda S_t = I_t$.

We can present equation (1a) as:

$$
K_{t+1} = K_t (1 - d) + \lambda_t S_t = K_t(1 - d) - (1 - \lambda_t)S_t + S_t
$$

Substituting (11) for $S_t$ in (1a) and substituting (14) for $(1 - \lambda_t)S_t$, we get:

$$
K_{t+1} = (1 - d)K_t - \frac{(\gamma_0 + \gamma_1)(L_{0t} + L_{1t})}{\alpha}K_t + \frac{\gamma}{1 + \gamma}L_{0t}Y_t
$$

Substituting the production function (3) into (15) we get:

$$
K_{t+1} = (1 - d)K_t - \frac{(\gamma_0 + \gamma_1)(L_{0t} + L_{1t})}{\alpha}K_t + \frac{\gamma}{1 + \gamma}L_{0t}K_t^\alpha L_{1t}L_{2t}E_{0t}E_{1t}^{\gamma_1}E_{2t}^{\gamma_2}
$$

### Population Growth Rate

Let us assume that population growth rate of each class is defined as:

$$
X_n = n_0 - \sigma E_{1t} - \sum_{i=1}^{N} \delta_i Z_{it-1}
$$

For $n_0$ - an initially high growth rate, $\sigma$ - a depreciation factor in population growth due to education acquired and $Z_{it-1}$ - factors other than education affecting the population growth rate.

### Economic Growth

Production is determined by education level, population size and capital amount. Whenever the stock of capital grows, new firms will be created. This is due to the assumption that the optimal size of each firm will remain constant in all generations and no technological changes occur (long run equilibrium in production, which determines an optimal size of a firm in each generation is also assumed). The firms optimal amount of capital in all generations would be $k_t$ for

$$
k_t = \frac{K_t}{L_{0,t}} = \frac{K_{t+1}}{L_{0,t+1}} = \ldots = \frac{K_{t+k}}{L_{0,t+k}}, \text{ where } L_{0,t+k} \text{ is the number of firm owners in period } t + j.
$$

---

6 Notice that if there were no credit constraint over class 2, their education level would be $E_{2t} = \frac{\gamma_2 k_t}{\alpha}$.

7 $\lambda_t S_t = -(1 - \lambda_t)S_t + S_t$
In each generation, when the cumulative amount of capital grows, the number of firms grows. Given that each individual can own only one firm, we would get that the number of firm owners in each generation \( t + j \) would be \( L_{0,t+j} = \frac{K_{0,t}}{k_t} \) for \( j = 0,1,2,\ldots \).

The assumption that one individual cannot be an owner of more than one firm lies in the idea that risk management of savings would encourage the older generation to spread the loans (their savings) among the new entrepreneurs.\(^8\)

Given that the characteristics needed for being a good entrepreneur are distributed among the population, we get that in each generation there would be mobility of individuals \(^9\) from workers belonging to class 1 into the firm owners' class.

In each generation, we get that the number of firms and the number of firm owners is determined by the level of capital.

If capital level grows, then the number of firms and the number of firm owners grow.

Since population grows according to equation (16), we will get a lower population growth rate for classes 0 and 1 and a higher population growth rate for class 2.

In each time period, whenever capital grows, we get a change in income distribution.

Production growth, population growth and income distribution are determined simultaneously.

### 3. Simulation

Our aim in this part is to present an example that demonstrates how the suggested theoretical model can be applied for calculating income distribution along the growth path of the economy.

Let us define arbitrary parameters of the utility and of the production functions and determine an arbitrary value for the optimal stock of capital in each firm. We also assume that the tax rate is zero. Given a starting value of the aggregate capital in the economy, we can find the number of firms and the income distribution (by using the Gini index). In addition, we can calculate the growth path of capital stock and the changes in income distribution and population distribution along this path. Given that the optimal size of a firm stays constant in each generation, the number of firms will increase at the same rate as the capital stock. An increase in the number of firms would first be filled by the natural population growth of agents of class zero and then by agents from class one. Agents from class two, who cannot borrow, will not have the ability to move into another class, due to lack of education or capital.

The example will enable us to show that any change in the capital stock would lead to a change of income and population distribution.

#### The Production Function

Let assume that the production function of each firm \( j \) is:

\[
Y^j_t = 6K^j_t^{0.49}L^j_{1t}^{0.4}L^j_{2t}^{0.1}E^j_{1t}^{0.01}
\]

Let assume that the starting values in period \( t \) (current period) are:

\[
L_{0,t} = 100 \\
L_{1t} = 79, L_{2t} = 20
\]

\(^8\)We assume for simplicity’s sake that the optimal loan is equal to the capital needed for one firm, although the optimal loan might be larger or smaller. Change in the size of the optimal loan might change the quantitative, but not the qualitative results.

\(^9\)Any individual with initiative from class 1 can try to exploit the opportunity to become a firm owner. S/He also has to convince lenders to lend him/her money for financing the required amount of capital.
There are 100 firms and 100 firm owners. There are 2 categories of workers, in class one 7,900 workers and in class two 2,000 workers. Each firm employs 79 workers from class one and 20 worker from class two. Each firm employs an amount of 5 units of capital (the optimal loan for lenders), and the total amount of capital is 500. The initial education level of agents in class one is indexed to 0.1, so each firm that employs 79 workers from class one uses 7.9 units of education (while assuming a constant education level for agents belonging to class zero and two, which is not relevant to production process\(^{10}\)). The education level of agents from class zero is implied in production function technology and does not appear directly as a factor effecting productivity.

Under these conditions the product of each firm is:

\[
Y^j_t = 6K^j_t 0.49 L^j_t -0.6 L^j_{2t} 0.1 E^j_t 0.01 = 6*5^{0.49} * 79^{0.4} * 20^{0.1} * 7.9^{0.01} = 104.4183 \tag{3c}
\]

The aggregate product of the economy is:

\[
Y_t = L_{0j} Y^j_t = 100*104.4183 = 10441.83
\]

Where \( L_{0j} \) is the number of firms.

### Income

Each worker will get his/her marginal product according to equation (5a):

\[
w^j_t = 0.4*6K^j_t 0.49 L^j_t -0.6 L^j_{2t} 0.1 E^j_t 0.01 + 0.01*6K^j_t 0.49 L^j_t 0.4 L^j_{2t} 0.1 E^j_t -0.99 = 0.4*6*5^{0.49} * 79^{0.4} * 20^{0.1} * 7.9^{0.01} + 0.01*6*5^{0.49} * 79^{0.4} * 20^{0.1} * 7.9^{-0.99} = 0.660875
\]

\[
w^j_{2t} = 0.1*6*5^{0.49} * 79^{0.4} * 20^{0.1} * 7.9^{0.01} = 0.522092
\]

Total wages a firm pays to workers from class one is:

\[
w^j_{1t} = 0.4*6K^j_t 0.49 L^j_t -0.6 L^j_{2t} 0.1 E^j_t 0.01 + 0.01*6K^j_t 0.49 L^j_t 0.4 L^j_{2t} 0.1 E^j_t -0.99 = 0.4*6*5^{0.49} * 79^{0.4} * 20^{0.1} * 7.9^{0.01} + 0.01*6*5^{0.49} * 79^{0.4} * 20^{0.1} * 7.9^{-0.99} = 42.81151
\]

While total wages paid to workers from class 2 is: 0.522092 * 20 = 10.44183

Each firm owner will get the total firm product minus labor income:

\[
w^j_{0j} = (1-\beta_1-\gamma_1-\beta_2) Y^j_t = (1-0.4-0.01-0.1)*104.4183 = 51.16497
\]

### Income distribution

The total income of the lowest income class of workers is \( W_{2j} = \beta_2 Y_t = 0.1*Y_t \),

The income of workers from class one is \( W_{1j+1} = (\beta_1+\gamma_1) Y_{t+1} = 0.5Y_{t+1} \), while the income of the firm owners class is \( W_{0j} = (1-\beta_1-\gamma_1-\beta_2) Y_t = 0.4Y_t \).

Our main concern is to examine the distribution of income between an agent in class 2 relative to agents belonging to classes 0 and 1, which at initial period together get 0.9\( Y_t \).

---

\(^{10}\)For simplicity's sake, we assume that firm owners do not invest in education, but only in capital.
Table 1 presents income distribution

<table>
<thead>
<tr>
<th>Percentile</th>
<th>Total income (percent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20%</td>
<td>10%</td>
</tr>
<tr>
<td>100</td>
<td>100%</td>
</tr>
</tbody>
</table>

The Lorenz curve for this income distribution is presented in figure 1.

Figure 1

The Gini index for this income distribution is equal to 0.08.

Savings, Investment and Capital stock

Let assume that each individual, i, in the economy has the same utility function:

\[
U = \ln(C_{0i}) + 0.5 \ln(C_{1i})
\]

and that s/he maximized the utility subject to budget constraints:

\[
\text{Max} \quad \ln(C_{0i}) + 0.5 \ln(C_{1i})
\]

\[
\text{s.t} \quad C_{0i} + \frac{C_{1i}}{1+r} = W_i \ (1-t)
\]

We get that the savings of each individual is:

\[
S_i^t = \frac{0.5}{1+0.5} W_i .
\]

(7b)

The aggregate saving is:

\[
S_t = L_{0t} \sum_{i=0}^{2} S_i^t L_i = L_{0t} \frac{0.5}{1+0.5} \left[ \sum_{i=0}^{2} W_i L_i \right] = L_{0t} \frac{0.5}{1+0.5} Y
\]

(10b)

(We should remember that \( L_{0t} \) is the number of firms and the number of firm owners as well).

In the current generation, the aggregate saving is:

\[
S_t = 10^{4} \frac{0.5}{1+0.5} Y = \frac{0.5}{1+0.5} 10441.83 = 3480.61
\]

The level of education of workers in class 1 is

\[
E_{1t} = \frac{\gamma_i k_i}{\alpha} = \frac{0.01 \times 5}{0.49} = 0.102041.
\]

Assuming that the depreciation rate is 10%, the capital stock in the next generation will be:
The Number of Firms in Period \( t+1 \)

Since the capital stock in each firm is fixed and equal to 100, we get that the number of firms in each period would be \( L_{0,t+1} = \frac{K_{t+1}}{K_{t+1} - 1} \). In period \( t+1 \), the number of firms is:

\[ L_{0,t+1} = \frac{2724.488}{500} \times 100 = 544.8976, \]

which reflects an increase of 461.1123 firms in comparison to its number in period \( t \).

The new entrepreneurs would come from the firm owner class and from class 1.

Population growth

Assuming that population growth rate of each class is defined as:

\[ X_{it} = n_0 - \sigma E_{it} - \sum_{i=1}^{N} \delta_{i} Z_{it-1}, \]

for \( n_0 = 5\% \) - an initially high population growth rate, \( \sigma = 0.4 \) - a depreciation factor of growth due to education acquired and \( \delta_i = 0, \) for \( i=1, 2, \ldots, N \) (which means that initially there are no changes in factors other than education effecting the population growth rate). We also assume that the natural population growth rate of firm owners is equal to that of class one.

We get that the number of workers belonging to class two will be 2000*1.05=2100

The number of workers belonging to class 1 is 7900*[1+0.05-0.4*0.102041] = 7972.551

and the number of people who are originally from class zero is 100*[1+0.05-0.4*0.102041] =100.9184.

However, in period \( t+1 \) there are 544.8976 firm owners, which means that 544.8976 -100.9184= 443.9792 new entrepreneurs move from class 1 into the firm owners’ class (new entrepreneurs come only from classes zero and one).

Table 2 presents population distribution in period \( t \) and \( t+1 \).

**Table 2: Population Distribution and Education in Economic Classes**

<table>
<thead>
<tr>
<th>Period</th>
<th>Firm owners</th>
<th>Class 1</th>
<th>Class 2</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100 (1.00%)</td>
<td>7900 (79.00%)</td>
<td>2000 (20.00%)</td>
<td>10000 (100.00%)</td>
</tr>
<tr>
<td>( t+1 )</td>
<td>544.8976 (5.36%)</td>
<td>7528.572 (74.00%)</td>
<td>2100 (20.64%)</td>
<td>10173.47 (100.00%)</td>
</tr>
<tr>
<td>Employed per firm ( t+1 )</td>
<td>1</td>
<td>13.81649</td>
<td>3.853935</td>
<td></td>
</tr>
<tr>
<td>Education per firm</td>
<td>13.81649 * 0.102041 = 1.409846</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Following the change in the number of workers, we get a change in wages due to a change in marginal productivity and a change in income distribution represented by the Gini Index (See calculations in Appendix 2).

**Growth Path of the Economy**

The lowest income classes’ population grows faster than the two other classes, causing an increase in their relative portion of the total population.

Figure 2 presents the path of the relative size of class two.

![Figure 2](image)

In each generation, the total amount of capital grows and the number of firms grows\(^{11}\).

The wage of agents in each class is affected by three factors: 1) the population growth rate of a class negatively affects the marginal product of workers belonging to the class.

2) Investment in education of a class reduces population growth rate. Since only classes zero and one invest in education, their population growth rate is lower. A smaller population growth, together with the positive effect of education on wages, has a positive effect over these classes’ wages. 3) Investment in capital leads to an increase in the number of firms, causing an increase in the wages of classes two and one and a decrease in wages of class zero (the marginal product of labor increases while the marginal product of capital decreases).

We should note that wages of the lowest income class might increase in absolute terms, although the population of that class is growing fast. That is due to the effect of capital increase. However, their relative income becomes lower.

Figures 3 and 4 present the wage per worker in class two and the Gini index path of the economy.

\(^{11}\)According to the assumption of an optimal amount of capital \(k\), in each firm.
As we can see in appendix 3, class 2 wage per worker increases in some periods and decreases in other periods. However, the Gini index shows that income distribution worsens as the economy evolves. We can also see in figure 3 that class 1 wage per worker becomes higher than income per entrepreneur. That is because an increase in the total amount of capital increases the number of firms faster than the increase in class 1’s population growth rate, leading to a reduction in the number of workers employed per firm in classes 1 and 2, so that the marginal product (of labor) increases while the marginal product of capital decreases.

4. Policy Needed For Improving Income and Population Distribution
Population growth rate is defined as \( X_i = n_i - \sigma E_{i,t} - \sum_{i=1}^{n} \delta_{i}Z_{i-1} \), where \( Z_{i-1} \) are changes in factors such as culture, religion, government resources addressed for assistance etc., affecting the population growth rate.

If a government uses a tax policy that enables the allocation of resources in order to change \( Z_{i-1} \) and cause a reduction in \( X_{2r} \), we would get a change in the distribution of population and income along the growth path of the economy.
Simulation
Let us assume \( X_t = n_0 - \sigma, E_t, -\delta, Z_t \) with \( \sigma_0 = \sigma_1 = 0.4, E_{2,t} = 0, \delta_0 = \delta_1 = 0, \delta_2 = 0.1 \) \( Z_{2,t} = 0 \).

Using initial values assumed in the simulation presented in part IV above, we tax the income of classes 0 and 1 and use these taxes to effect \( Z_{2,t} \) in order to reduce the population growth rate of class 2.

The income of classes zero and one are taxed at a rate of \( t = 5\% \), where \( T \) is total taxes. All tax revenue is used for increasing \( Z_{2,t} \).

Let assume that: \((16b)\) \( \delta Z_{2} = \delta_2 \frac{T_t}{L_{2,t}} = 0.04 \frac{T_t}{L_{2,t}} \).

Figures 5, 6 and 7 present the path of the relative size of class 2, Gini index, and production level, before and after government intervention, respectively.
It can easily be seen that this government policy would improve income distribution and wage per worker from class 2 while decreasing the relative size of the poor class along the growth path of the economy.

We should note that improving income distribution is accompanied by a reduction in productivity along the growth path. (We assume that resources are removed from the economy and transferred solely for the purpose of reducing the class 2 population growth rate\(^\text{12}\)).

We can see that the wage per worker belonging to class 2 is much higher than that before applying the suggested government program.

\(^{12}\)In reality, the measurement of production includes services such as awareness programs, medical advisory or any other means that would lead to increasing the Z factor. This means that applying such a program will not necessarily reduce production, but only change income and population distribution.
5. Summary

The main aim of this paper is to suggest a theoretical framework for analyzing the existence of different population growth rates, for different income classes, on population and income distribution along the growth path of the economy.

Using the framework of an overlapping generation model, savings finance investment in capital and education. The poorest class, class 2, is not able to invest in capital or education due to credit constraints. The other two classes, i.e. class zero (firm owners) and class one (higher class workers) invest in capital and in education.

The population growth rate is affected by education and other factors which are not specified here. Given a higher education for the richer classes, their population growth rate is lower. As the economy evolves, the relative size of the poorest population increases and its wage per worker decreases due to a reduction in its marginal product. Assuming that the optimal size of a firm is given (according to risk considerations of the lenders), as the cumulative amount of capital increases, the number of firms and the number of firm owners increase, leading to the mobility of some of the highest income classes’ workers into the firm owner class. A fast increase in the number of firms reduces the number of workers per firm, leading to a change in the wage per worker of the two working classes and a change in the firm owners’ income. We show an example with a specific production function for a firm that uses labor capital and education. Given the equalization of the marginal product of capital and education, the level of education for the two highest income classes is determined. Education is financed by borrowing from the previous generation. The rest of the previous generation's savings finance new acquired capital. For each generation, we can calculate production, capital, savings, education, wages per worker of each class, mobility of workers from a higher income class into the firm owners' class and the income distribution.

We present a simulation that uses initial levels of capital and labor in each class and show the change in the distribution of income and population along the growth path of the economy. The simulation demonstrates that the relative size of the poorest class would increase, the wage per worker from the lowest income class would decrease and income distribution would become worse as time passes. We present one possible government policy that taxes the highest income classes and uses that tax revenue in order to reduce the population growth rate of the lowest income class. Such a scheme would cause a reduction in the relative size of the poorest class, increase the wage per worker of the poorest class and improve income distribution.

Our intention is not to suggest a specific scheme for reducing the population growth rate of the lowest income class, but only to point out that allocation of resources for that goal would have positive economic and social benefits. A specific scheme of how to reduce the population growth rate should be examined in future research (together with researchers from various relevant disciplines).

### Appendix 1 – World Population

<table>
<thead>
<tr>
<th>Year</th>
<th>1750</th>
<th>1800</th>
<th>1850</th>
<th>1900</th>
<th>1950</th>
<th>1999</th>
<th>2050</th>
<th>2150</th>
</tr>
</thead>
<tbody>
<tr>
<td>World</td>
<td>791</td>
<td>978</td>
<td>1262</td>
<td>1650</td>
<td>2522</td>
<td>5978</td>
<td>8909</td>
<td>9747</td>
</tr>
<tr>
<td>Africa</td>
<td>106</td>
<td>107</td>
<td>111</td>
<td>133</td>
<td>221</td>
<td>767</td>
<td>1766</td>
<td>2308</td>
</tr>
<tr>
<td>Asia</td>
<td>502</td>
<td>635</td>
<td>809</td>
<td>947</td>
<td>1402</td>
<td>3634</td>
<td>5268</td>
<td>5561</td>
</tr>
<tr>
<td>Europe</td>
<td>163</td>
<td>203</td>
<td>276</td>
<td>408</td>
<td>547</td>
<td>729</td>
<td>628</td>
<td>517</td>
</tr>
<tr>
<td>Latin America</td>
<td>16</td>
<td>24</td>
<td>38</td>
<td>74</td>
<td>167</td>
<td>511</td>
<td>809</td>
<td>912</td>
</tr>
<tr>
<td>North America</td>
<td>2</td>
<td>7</td>
<td>26</td>
<td>82</td>
<td>172</td>
<td>307</td>
<td>392</td>
<td>398</td>
</tr>
<tr>
<td>Oceania</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>6</td>
<td>13</td>
<td>30</td>
<td>46</td>
<td>51</td>
</tr>
</tbody>
</table>
Appendix 2 – Variables in Period t+1

Firms’ production is:
\[ Y^j_{t+1} = 6K^j_{t+1}^{0.49} L^j_{t+1}^{0.4} L^j_{2t+1}^{0.1} E^j_{t+1}^{0.01} = 6 * 5^{0.49} * 13.81649^{0.4} * 3.853935^{0.1} * (0.102041*13.81649)^{0.01} = 43.33989 \]

The aggregate product of the economy is:
\[ Y_{t+1} = L_{t+1}^j Y^j_{t+1} = 544.8976*43.33989 = 23615.8 \]

Where \( L_{t+1}^j \) is the number of firms.

Income

Each worker will get his/her marginal product according to equation (5a):
\[
w_{t+1} = 0.4 * 6K_{t+1}^{0.49} L_{t+1}^{0.4} L_{2t+1}^{0.1} E_{t+1}^{0.01} + 0.01 * 6K_{t+1}^{0.49} L_{t+1}^{0.4} L_{2t+1}^{0.1} E_{t+1}^{0.99} = \\
0.4 * 5^{0.49} * 13.81649^{0.4} * 3.853935^{0.1} * (0.102 * 13.81649)^{0.01} + 0.01 * 5^{0.49} * 13.81649^{0.4} * 3.853935^{0.1} * (0.102 * 13.81649)^{0.99} = 1.562 \]

\[ w_{2t} = 0.1 * 5^{0.49} * 13.81649^{0.4} * 3.853935^{0.1} * (0.102 * 13.81649)^{0.01} = 1.124562 \]

Total wages a firm pays to workers from class one is:
\[
w_{t+1} = 0.4 * 6K_{t+1}^{0.49} L_{t+1}^{0.4} L_{2t+1}^{0.1} E_{t+1}^{0.01} + 0.01 * 6K_{t+1}^{0.49} L_{t+1}^{0.4} L_{2t+1}^{0.1} E_{t+1}^{0.99} = \\
0.4 * 5^{0.49} * 13.81649^{0.4} * 3.853935^{0.1} * (0.102 * 13.81649)^{0.01} + 0.01 * 5^{0.49} * 13.81649^{0.4} * 3.853935^{0.1} * (0.102 * 13.81649)^{0.99} = 17.76935 \]

while total wages paid to workers from class 2 is: \( 1.124562*3.853935 = 4.333989 \)

Each firm owner will get the total firm product minus labor income:
\[
W'_{0,t} = (1 - \beta_1 - \gamma_1 - \beta_2)Y^j_t = (1 - 0.4 - 0.1 - 0.1) * 43.33989 = 21.23655 \]
Income distribution
Each class will get a constant share of production, class 2 will get $W_{2, t+1} = \beta_2 Y_{t+1} = 0.1 Y_{t+1}$, class 2 will get $W_{2, t+1} = (\beta_1 + \gamma_1) Y_{t+1} = 0.5 Y_{t+1}$ and class 0 will get $W_{0, t+1} = (1-\beta_1-\gamma_1) Y_{t+1} = 0.4 Y_{t+1}$.

The number of workers belonging to class 1 is $7900 *[1+0.05-0.4*0.102041) = 7972.551$ and the number of people which are originally from class zero is $100*[1+0.05-0.4*0.102041) = 100.9184$.

However, in period t+1 there are $544.8976$ firm owners, which means that $544.8976 -100.9184 = 443.9792$ new entrepreneurs move from class 1 into the firm owners’ class.

Table 2 presents population distribution in period t and t+1.

Our main concern is to examine the distribution of income between agents in class 2 relative to agents belonging to classes 0 and 1, which at initial period together get $0.9 Y_t$.

Table B present income distribution:

<table>
<thead>
<tr>
<th>Percentile</th>
<th>Total income (percent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.64%</td>
<td>10%</td>
</tr>
<tr>
<td>100</td>
<td>100%</td>
</tr>
</tbody>
</table>

Gini index for this income distribution is 0.085777.

References


N. B. David & U. B. Zion


